Int. J. Solids Structures Vol. 14, pp. 527–533
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0020-7683/78/0601-0527/$02.00/0

THE ELASTIC INSTABILITY OF A DIRECTOR ROD

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(*Received* 28 *October* 1977; *received for publication* 15 *February* 1978)

Abstract—Instability of a linear dipolar rod, or Timoshenko beam, is treated without uncoupling the pair of governing partial differential equations. It is shown by modified logarithmic arguments, that at and above the critical load both components of displacement grow in norm for large time. Useful information on post-buckling behaviour of the rod is thereby provided.

1. INTRODUCTION

One successful application of multipolar continuum mechanics has been to the director theory of rods. In this context, a general nonlinear rod theory has been developed by Green and Laws[1], and the associated linearised theory which contains as special case the Timoshenko beam theory, has been presented by Green, Knops and Laws[2]. The latter study also included a stability analysis of the straight rod under a large simple compression with respect to perturbations composed of small flexural motions. In terms of the elastic constants of the material, an upper bound was obtained on the applied compressive load in order that the straight position of the rod be stable.

The stability analysis was continued by Knops[3] who showed in what sense the previously derived upper bound may be regarded as critical. He proved, by means of logarithmic convexity arguments, that for certain initial data and for loads in excess of the bound the sum of the weighted L_2-norms of the displacement and director displacement eventually becomes at least exponentially unbounded in time. The straight position is then unstable so that the bound represents a critical load for stability.

Here, we extend slightly the instability analysis, but more significantly, investigate the precise mechanism by which the straight rod loses its stability. Specifically, we wish to answer the question: does both the displacement and director displacement simultaneously become unbounded in norm, or does only one constituent grow? This behaviour is important for a knowledge of how the rod together with its cross section deforms during the post-buckling process. Additionally, the methods developed represent a new application of logarithmic convexity arguments to multi-component systems. We shall prove that above the critical load, the L_2-norm of the displacement becomes at least exponentially unbounded with increasing time, while the director displacement becomes similarly unbounded for loads which, although exceeding the critical one, are themselves not too large. At the critical load, growth is sustained, but now it is quadratic.

In Section 2 we set down the basic theory and briefly recount known stability results as well as deriving a few new results about behaviour at the critical load. Section 3 establishes separate growth estimates for the displacement and director displacement based upon logarithmic convexity arguments. Finally, in Section 4 we demonstrate how our analysis may be extended to other problems by considering the stability of an isotropic elastic plate subject to uniform compression in the class of small flexural motions. Throughout, we assume existence of classical solutions on the whole time interval.

2. BASIC THEORY AND STABILITY RESULTS

Small superposed one-dimensional flexural deformations of a straight elastic rod subjected to a large simple compression are governed by the coupled pair of partial differential equations (see Green, Knops and Laws[2])

$$(1-\xi^2)\frac{\partial^2 v}{\partial x^2}+\frac{\partial b}{\partial x}=m\frac{\partial^2 v}{\partial t^2}, \tag{1}$$

$$\xi\frac{\partial^2 b}{\partial x^2}-b-\frac{\partial v}{\partial x}=n\frac{\partial^2 b}{\partial t^2}, \tag{2}$$

where $x\in[0,1]$ is the non-dimensionalised spatial coordinate defining points on the rod, t (≥ 0) denotes the time variable, $v(x, t)$ is the non-dimensionalised displacement of the rod perpendicular to the rod and $b(x, t)$, the relevant director displacement, is a measure of the shear in the rod along the rod. Also, ξ, m, n are positive constants depending upon the material and rod length, and λ^2 is a positive parameter proportional to the compressive stress in the rod.

We examine the stability and instability of the null solution $v\equiv 0$, $b\equiv 0$ to eqns (1) and (2) subject to the ends of the rod being both either simple-supported or clamped. The appropriate boundary conditions are respectively:

$$v(0,t)=\frac{\partial b}{\partial x}(0,t)=0,\qquad v(1,t)=\frac{\partial b}{\partial x}(1,t)=0,\ t\geq 0, \tag{3}$$

$$v(0,t)=b(0,t)=0,\qquad v(1,t)=b(1,t)=0,\ t\geq 0. \tag{4}$$

Initial conditions are taken to be

$$v(x,0)=v_0(x),\qquad b(x,0)=b_0(x),\qquad x\in[0,1] \tag{5}$$

$$\frac{\partial v}{\partial t}(x,0)=\dot v_0(x),\qquad \frac{\partial b}{\partial t}(x,0)=\dot b_0(x),\qquad x\in[0,1] \tag{6}$$

where v_0, b_0, $\dot v_0$, $\dot b_0$ are prescribed functions.

We define the kinetic and potential energies respectively by

$$K(t)=\frac{1}{2}\int_0^1\left[m\left(\frac{\partial v}{\partial t}\right)^2+n\left(\frac{\partial b}{\partial t}\right)^2\right]\mathrm{d}x, \tag{7}$$

$$V(t)=\frac{1}{2}\int_0^1\left\{\xi\left(\frac{\partial b}{\partial x}\right)^2+b^2+(1-\xi\lambda^2)\left(\frac{\partial v}{\partial x}\right)^2+2b\frac{\partial v}{\partial x}\right\}\mathrm{d}x. \tag{8}$$

It immediately follows from (1) and (2) by integration combined with use of (3) and (4) that total energy of the system is conserved:

$$E(t)\equiv K(t)+V(t)=E(0). \tag{9}$$

A straightforward calculation based on *a priori* inequalities [2, eqn 7.18] shows that the potential energy is positive-definite provided $\lambda^2<k^2$, where

$$k^2=\pi^2/(1+\xi\pi^2) \tag{10}$$

for simply-supported ends and

$$k^2=4\pi^2/(1+4\xi\pi^2) \tag{11}$$

for clamped ends. On taking $E(t)$ as Liapunov function, stability may be established easily for $\lambda^2<k^2$ with respect to the measure

$$\sup_{t>0}\max_{x\in[0,1]}\,[|v(x,t)|+|b(x,t)|], \tag{12}$$

and the initial measure $E(0)$. Details may be found in Green, Knops and Laws [2].

To derive uniqueness of the solution to (1) and (2) together with some preliminary growth estimates we employ logarithmic convexity arguments and for later reference briefly repeat a procedure described by Knops [3] (see also Knops and Payne [5]). Thus, consider the function

$F(t)$ defined by

$$F(t)=\int_0^1 (mv^2+nb^2)\,\mathrm{d}x+\alpha(t+t_0)^2, \tag{13}$$

where α, t_0 are positive constants to be subsequently determined. Differentiation of (13) leads to

$$\frac{\mathrm{d}F}{\mathrm{d}t}(t)=2\int_0^1 \left(mv\frac{\partial v}{\partial t}+nb\frac{\partial b}{\partial t}\right)\mathrm{d}x+2\alpha(t+t_0), \tag{14}$$

$$\frac{\mathrm{d}^2F}{\mathrm{d}t^2}(t)=2\int_0^1 \left[m\left(\frac{\partial v}{\partial t}\right)^2+n\left(\frac{\partial b}{\partial t}\right)^2\right]\mathrm{d}x+2\int_0^1 \left[mv\frac{\partial^2 v}{\partial t^2}+nb\frac{\partial^2 b}{\partial t^2}\right]\mathrm{d}x+2\alpha. \tag{15}$$

On substituting in (15) for the inertia terms from (1) and (2) and integrating by parts, we obtain

$$\frac{\mathrm{d}^2F}{\mathrm{d}t^2}(t)=4K(t)-4V(t)+2\alpha \tag{16}$$

$$=8K(t)-4E(0)+2\alpha, \tag{17}$$

where conservation of energy (9) has been used to derive (17). Next, we take (14) and (17) and apply Schwarz's inequality to readily derive the following inequality:

$$F\frac{\mathrm{d}^2F}{\mathrm{d}t^2}-\left(\frac{\mathrm{d}F}{\mathrm{d}t}\right)^2\geqslant -2(\alpha+2E(0))F(t). \tag{18}$$

To prove uniqueness it must be demonstrated that only the null solution exists to (1), (2) subject to homogeneous initial data (5) and (6). But then $E(0)=0$ and so on setting $\alpha=0$, the right side of (18) vanishes. Let us suppose the solution is non-unique, so that $F(t)>0$ for $t\in(t_1, t_2)$. Then from (18) we have

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\ln F(t))\geqslant 0, \qquad t\in(t_1,t_2), \tag{19}$$

which on integration gives

$$F(t)\leqslant [F(t_1)]^{t_2-t/t_2-t_1}[F(t_2)]^{t-t_1/t_2-t_1}, \qquad t\in(t_1,t_2). \tag{20}$$

By continuity $F(t_1)=0$. Therefore, we may conclude from (20) that $F(t)=0$, $t\in[t_1,t_2]$, contrary to hypothesis and so uniqueness is established. We note that the proof does not rely upon any sign-definiteness assumptions and, in particular, remains valid even when ξ, m, n, λ^2 are allowed to become negative. Furthermore, suppose $F(t)$ vanishes at some instant t_3, and that $E(0)\leqslant 0$. Then, we may reverse the time direction in the above argument to show that $F(t)$ vanishes for all time. Thus, in what follows, we are justified in assuming for $E(0)\leqslant 0$ and non-zero initial data that $F(t)$ never vanishes for any t. To treat instability, we first impose the restriction $\lambda^2>k^2$; the case $\lambda^2=k^2$ will be discussed later in this section. The potential energy ceases to be positive-definite and hence initial data can be chosen to make $E(0)<0$. We select α to satisfy $\alpha+2E(0)\leqslant 0$. By hypothesis, $F(t)$ is always strictly positive, so that inequality (18) may be written in the form (19), which as before may be integrated to give an inequality of type (20). From this we may deduce Hölder continuity of the solution upon its initial data on compact sub-intervals of $[0, T]$ in the class of solutions for which $F(T)$ is bounded. An alternative integration of (19) yields

$$F(t)\geqslant F(0)\exp\left[\frac{t\,\mathrm{d}F}{\mathrm{d}t}(0)/F(0)\right]. \tag{21}$$

Irrespective of the initial data, $\mathrm{d}F(0)/\mathrm{d}t$ may be made always positive by suitable choice of t_0.

Consequently, we conclude from (21) that the sum of the weighted L_2-norms of the displacement and director displacement increases at least exponentially for sufficiently large times.

A similar estimate may be obtained for the kinetic energy (7). We integrate (19) once to obtain

$$\frac{F(t)}{F(0)}\frac{dF(0)}{dt} \le \frac{dF}{dt}(t), \tag{22}$$

and apply Schwarz's inequality to get after setting $\alpha = 0$,

$$\frac{G(t)}{G^2(0)}\frac{dG(0)}{dt} \le 8K(t), \tag{23}$$

where

$$G(t) = \int_0^1 (mv^2 + nb^2)\, dx. \tag{24}$$

Thus, the kinetic energy has the same growth behaviour as $F(t)$. In particular, when the initial data is such that $E(0) < 0$, $K(t)$ increases at least exponentially for sufficiently large time.

We next examine the case $\lambda^2 = k^2$. From inequalities given in [1, eqn 7.18] it follows that the potential energy is positive semi-definite ($V(t) \ge 0$, $t \ge 0$) and therefore

$$K(t) \le E(0). \tag{25}$$

We deduce from (5) and (17) with $\alpha = 0$ that

$$\frac{d^2G(t)}{dt^2} \le 4E(0) \tag{26}$$

and so

$$G(t) \le G(0) + t\frac{dG(0)}{dt} + 2t^2E(0), \tag{27}$$

which shows that the constituent displacements in L_2-norm possess at most quadratic growth in time. We shall identify conditions when there is precisely quadratic growth.

Select initial data to make $V(0) = 0$, so that $v_0(x)$ and $b_0(x)$ are solutions to the associated equilibrium problem. Furthermore suppose

$$\frac{dG}{dt}(0) > 0, \qquad \left[\frac{dG}{dt}(0)\right]^2 = 8K(0)G(0). \tag{28}$$

By continuity, there exists an instant t_1 such that

$$\frac{dG}{dt} > 0, \qquad t \in [0, t_1). \tag{29}$$

Set $\alpha = 0$ in (2.18), multiply by $G^{-3}(t)\, dG(t)/dt$ and then integrate to get

$$\left[\frac{dG(t)}{dt}\Big/G(t)\right]^2 \ge 8K(0)/G(t), \qquad t \in [0, t_1]. \tag{30}$$

We therefore conclude that (28) holds on $[0, \infty)$ and so also does (30). A further integration of (30) therefore leads to

$$G(t) \ge G(0) + 2\sqrt{[E(0)G(0)]} + 2t^2G(0), \qquad t > 0 \tag{31}$$

which on recalling (27) shows that $G(t)$ has precisely quadratic growth. Furthermore, by means of Schwarz's inequality applied to (30) we have

$$K(0) \leq K(t) \tag{32}$$

which with (25) yields

$$K(t) = K(0), \quad V(t) = 0. \tag{33}$$

The identical vanishing of the potential energy means that $v(x, t)$ and $b(x, t)$ are proportional to the respective constituents $\bar{v}(x)$, $\bar{b}(x)$ in the associated equilibrium problem, and by substitution into (1) and (2), it easily follows that

$$v(x, t) = c_1\bar{v}(x)(1 + t), \qquad b(x, t) = c_2\bar{b}(x)(1 + t), \tag{34}$$

where c_1, c_2 are constants. (Recall that nontrivial $\bar{v}$, $\bar{b}$ exist since $\lambda^2 = k^2$).

Thus, under initial data (28) together with $V(0) = 0$ we have obtained the complete solution to our problem when $\lambda^2 = k^2$, and we can see from (34) that both constituents grow in norm. The same information in the case $\lambda^2 > k^2$ cannot be derived from (21) which gives growth only of the sum of the respective norms. It is therefore natural to explore growth behaviour in these other cases for each constituent separately and this is carried out in the next section. It is finally worth remarking that the combined results of this section establish that $\lambda^2 < k^2$ is necessary for the stability of the null solution with respect to the measures $E(0)$ and (8).

3. CONSTITUENT INSTABILITY

In the previous section, we saw that above the critical load initial data may be chosen such that the displacement and director displacement become unbounded in the aggregate norm (24) for sufficiently large time. Clearly, this behaviour may be caused by growth of either one or both constituent displacements, or by each having oscillations of increasing amplitude exactly out of phase. The precise nature of the growth pattern is important when investigating post-buckling behaviour of the director rod, since it is of obvious interest to discover whether, for instance, the rod buckles without excessive longitudinal shear (represented by $b(x, t)$ remaining bounded.) A farily complete analysis of this question has just been given for the critical load ($\lambda^2 = k^2$), and here we wish to investigate what happens when the load is above its critical value ($\lambda^2 > k^2$). We prove, by methods applicable to more complex multi-component problems, that in general both displacements grow. In the next section, we briefly extend the analysis to plates.

Let us suppose, therefore, that initial data is chosen such that $G(t)$, given by (24), becomes unbounded in time. When $E(0) \leq 0$, (33) implies that the kinetic energy $K(t)$ possesses the same behaviour. By using a similar argument, this conclusion may be proved also to hold in the case $E(0) > 0$, $V(0) \leq 0$, $(\mathrm{d}G(0)/\mathrm{d}t) > 0$.

Let us now consider the conservation of energy (9). Completion of the square with respect to b and $\partial v/\partial x$ in the potential energy (8) at once leads to

$$K(t) - \xi\lambda^2 \int_0^1 \left(\frac{\partial v}{\partial x}\right)^2 \mathrm{d}x \leq E(0), \tag{35}$$

and hence it follows that

$$\int_0^1 \left(\frac{\partial v}{\partial x}\right)^2 \mathrm{d}x \tag{36}$$

becomes unbounded with $G(t)$. We next separately consider the cases (a) $\xi k^2 < \xi\lambda^2 < 1$ and (b) $\xi\lambda^2 \geq 1$.

(a) $\xi k^2 < \xi\lambda^2 < 1$

We apply the arithmetic-geometric mean inequality to the potential energy and obtain from the conservation of energy (9) the inequality

$$K(t) \leq \frac{1}{(1-\xi\lambda^2)} \int_0^1 b^2 \,\mathrm{d}x + E(0); \tag{37}$$

thus the L_2-norm of the director displacement has the same growth as $G(t)$.

Integration by parts of the cross term in the potential energy followed by another application of the arithmetic-geometric mean inequality enables us to derive from (9) a second inequality

$$K(t) + (1-\xi\lambda^2) \int_0^1 \left(\frac{\partial v}{\partial x}\right)^2 \mathrm{d}x \leq \frac{1}{\xi} \int_0^1 v^2 \,\mathrm{d}x + E(0), \tag{38}$$

and consequently, the L_2-norm of the displacement likewise increases as $G(t)$.

(b) $\xi\lambda^2 \geq 1$

We introduce the auxiliary function $J(t)$ defined by

$$J(t) = 3 \int_0^1 mv^2 \,\mathrm{d}x - \int_0^1 nb^2 \,\mathrm{d}x. \tag{39}$$

On differentiating twice, using eqns (1) and (2) together with an integration by parts and conservation of energy (9), we obtain

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}t^2} J(t) &= -4E(0) + 4\int_0^1 \left\{2m\left(\frac{\partial v^2}{\partial t}\right) + \xi\left(\frac{\partial b}{\partial x}\right)^2 + b^2 - (1-\xi\lambda^2)\left(\frac{\partial v}{\partial x}\right)^2\right\} \mathrm{d}x. \\ &\geq -4E(0). \end{aligned} \tag{40}$$

Integration immediately yields

$$J(t) \geq J(0) + t\frac{\mathrm{d}J}{\mathrm{d}t}(0) - 2t^2 E(0), \tag{41}$$

and hence addition of (21), (or its equivalent when $E(0) > 0$, $V(0) \leq 0$, $(\mathrm{d}G/\mathrm{d}t)(0) > 0$) and (41) shows that the L_2 norm of the displacement increases with $G(t)$.

It has thus been proved that in both cases, the norm of the displacement becomes unbounded with $G(t)$. Also, in the first case, that is when the compressive load is not too large, the director displacement likewise becomes unbounded with $G(t)$. Whether there is growth of the director displacement when $\xi\lambda^2 \geq 1$ remains an open question.

4. SMALL DISTURBANCES OF AN ISOTROPIC ELASTIC PLATE UNDER UNIFORM COMPRESSION

Green and Naghdi[5] have discussed the theory of small deformations superimposed upon a large deformation for an elastic shell. In particular, they studied equations for the small motion of an isotropic elastic plate subjected to large extension in two perpendicular directions. They showed that for transverse flexural motions the equations of the plate can be written in the form

$$h^{\alpha\beta\lambda\mu} b_{\mu,\lambda\alpha} - h^{\alpha\beta}(b_\alpha + \mathrm{d}u_{,\alpha}) = \rho_0 j\left(\frac{\partial^2 b_\beta}{\partial t^2}\right) \tag{42}$$

$$\mathrm{d}h^{\alpha\beta}(b_{\beta,\alpha} + \mathrm{d}u_{,\beta\alpha}) - \xi^2 f^{\alpha\beta} u_{,\alpha\beta} = \rho_0 \left(\frac{\partial^2 u}{\partial t^2}\right) \tag{43}$$

where the summation convention and comma notation are employed, greek indices range over the set $\{1,2\}$, ρ_0, d, j, ξ are positive material constants, $h^{\alpha\beta\lambda\mu}$ is a positive definite, isotropic material fourth-order tensor and $h^{\alpha\beta}$ is a positive diagonal material tensor. Also $\xi^2 f^{\alpha\beta}$ is a positive diagonal tensor and represents the compressive loading of the plate. We consider these equations in a bounded regular region Ω of $\mathbf{R}^2$. It follows, under boundary conditions of either simply supported edges or clamped edges (see Green and Naghdi[5]) that there is again conservation of energy given by

$$\begin{aligned} E(t) &\equiv \frac{1}{2}\int_\Omega \left[\rho_0\left(\left(\frac{\partial u}{\partial t}\right)^2 + jb_\beta{}^2\right) + h^{\alpha\beta\lambda\mu}b_{\mu,\lambda}b_{\beta,\alpha} + h^{\alpha\beta}(b_\alpha + \mathrm{d}u_{,\alpha})(b_\alpha + \mathrm{d}u_{,\beta}) - \xi^2 f^{\alpha\beta}u_{,\alpha}u_{,\beta}\right]\mathrm{d}A \\ &= E(0), \end{aligned} \tag{44}$$

where $\mathrm{d}A$ denotes the element of area in the plate.

Green and Naghdi showed that the energy functional is positive definite provided ξ, and hence the applied load, is less than a maximum value θ. For $\xi < \theta$, they were then able to deduce in the usual way that the null solution is Liapunov stable with respect to norms $\int_\Omega u^2\,\mathrm{d}A$, $\int_\Omega b_\beta{}^2$ and initial norm $E(0)$. When $\xi \geq \theta$, the energy functional becomes non-positive definite and using logarithmic convexity, it is again easy to show that for negative initial total energy, the measure $F^*(t)$ defined by

$$F^*(t) = \int_\Omega \rho_0(u^2 + jb_\beta{}^2)\,\mathrm{d}A \tag{45}$$

is at least exponentially increasing for large time.

Up to this point we have not assumed that the plate is under uniform compression. If this assumption is now made, then

$$h^{\alpha\beta} = h\delta^{\alpha\beta} \tag{46}$$

$$f^{\alpha\beta} = f\delta^{\alpha\beta} \tag{47}$$

where h and f are positive scalars, and $\delta^{\alpha\beta}$ is the Kronecker delta. It is then easy to show, as in Section 3, that when $E(0) < 0$, the measure $\int_\Omega \rho_0 u^2\,\mathrm{d}A$ grows at least exponentially. Furthermore, provided $\mathrm{d}^2 h - \xi^2 f > 0$, the measure $\int_\Omega \rho_0 j b_\beta{}^2\,\mathrm{d}A$ also grows at least exponentially.

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